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THE BOYARSKY–MEYERS INEQUALITY FOR THE ZAREMBA PROBLEM FOR $p(\cdot)$-LAPLACIAN

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We study the higher integrability of solutions to the Zaremba problem for the $p(\cdot)$-Laplacian in a plane domain with Lipschitz boundary. We prove that the Boyarsky–Meyers estimates for solutions are valid under a special ratio between the parts of the Dirichlet and Neumann conditions on the boundary. Bibliography: 16 *titles.*

1 Introduction

The paper is devoted to the study of the higher integrability of the gradient of solutions to the Zaremba problem for the inhomogeneous $p(\cdot)$-Laplace equation in a Lipschitz domain $D \subset \mathbb{R}^2$ with a variable exponent p such that

$$1 < \alpha \leqslant p(x) \leqslant \beta < \infty \quad \text{for almost all } x \in D. \tag{1.1}$$

We introduce the function space

$$W(D) = \{v \in W^1_\alpha(D), |\nabla v|^{p(\cdot)} \in L_1(D)\} \tag{1.2}$$

with the Sobolev–Orlicz norm

$$\|v\|_{W^1_{p(\cdot)}(D)} = \|v\|_{L_\alpha(D)} + \|\nabla v\|_{L_{p(\cdot)}(D)}, \tag{1.3}$$

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REGULAR FUNDAMENTAL SOLUTION TO PARABOLIC EQUATION WITH DINI CONTINUOUS COEFFICIENTS IN MANY SPATIAL VARIABLES

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We consider a parabolic equation with Dini continuous coefficients in the case of many spatial variables The existence of a regular fundamental solution to this equation is proved, and estimates for the solution are obtained. These results imply that the Dini condition for the principal coefficients of the equation is sharp. Bibliography: 8 *titles.*

It is known that for a uniformly parabolic equation with Hölder coefficients there exists a regular fundamental solution constructed by the Lévy method (see, for example, [1, 2]).

The existence of a fundamental solution to a parabolic equation of an arbitrary order in one spatial variable with Dini continuous coefficients was proved in [3], where it is assumed that the corresponding modulus of continuity ω_0 satisfies the double Dini condition

$$\int\limits_0^z y^{-1}dy \int\limits_0^y \omega_0(\xi)\xi^{-1}d\xi < +\infty, \quad z > 0.$$

The existence of a regular fundamental solution to a second-order equation in many spatial variables was proved in [4] under the double Dini condition. All the cited works use the classical Lévy method.

On the other hand, from [5] it follows that if the Dini condition is not satisfied by the principal coefficients in the equation (see (1.1) below); namely, if the convergence condition is not satisfied by the corresponding integral, then a regular fundamental solution do not necessarily exist.

The question arises whether the Dini condition is sufficient for the existence of a regular

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MULTIFREQUENCY INVERSE PROBLEM OF SCALAR ACOUSTICS: REMARKS ON NONUNIQUENESS AND SOLUTION ALGORITHM

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A three-dimensional multifrequency inverse problem of acoustic sounding of a stationary inhomogeneous medium is considered. This nonlinear inverse problem is reduced to solving an auxiliary three-dimensional linear Fredholm integral equation of the first kind. In the analysis of the uniqueness of the solution to the inverse problem, the connection between the integral equation and determining the source in the Helmholtz equation is indicated. The last problem is ambiguously solvable in the general case. Examples of such ambiguity are given. Questions about detailed data (frequencies, sources) ensuring or not the uniqueness of solutions are considered. A speed-efficient algorithm for solving the inverse problem based on Fourier transforms is proposed. This algorithm makes it possible to calculate uniquely an approximate solution by a stable method under data perturbations. The results of numerical experiments on solving a three-dimensional model inverse problem on fairly detailed grids are presented. Bibliography: 23 *titles. Illustrations*: 2 *figures.*

1 Introduction

Let a scalar function $p(\mathbf{x}, t)$ define an acoustic wave field depending on the coordinates $\mathbf{x} \in \mathbb{R}^3$ and time $t \geqslant 0$. The field $p(\mathbf{x}, t)$ in an infinite medium is excited by sources localized in a known region $S \subset \mathbb{R}^3$. The medium is characterized by the local phase speed of sound $c(\mathbf{x}) \in C^1(\mathbb{R}^3)$, $c(\mathbf{x}) > 0$, and has a constant density. Moreover, it is known that $c(\mathbf{x}) = c_0 = \text{const} > 0$ outside

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INVERSE PROBLEMS FOR THE HIGHER ORDER NONLINEAR SCHRÖDINGER EQUATION

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The results on the well-posedness of inverse problems with integral overdetermination on a bounded interval for the higher order nonlinear Schrödinger equation are established. Either the right-hand side of the equation, or the boundary data, or both are chosen as controls. Assumptions on the smallness of the input data or a time interval are required. Bibliography: 28 *titles.*

1 Introduction. Notation. Description of the Main Results

In this paper, we study three inverse problems for the higher order nonlinear Schrödinger equation

$$iu_t + \alpha u_{xx} + ibu_x + iu_{xxx} + \lambda |u|^p u + i\beta(|u|^p u)_x + i\gamma(|u|^p)_x u = f(t,x) \tag{1.1}$$

on an interval $I = (0, R)$ ($R > 0$ is arbitrary). Here, α, β, γ, λ, b are real constants, $p \geqslant 1$, $u = u(t,x)$ and f are complex-valued functions.

We consider the initial-boundary value problem for Equation (1.1) in the rectangle $Q_T = (0,T) \times I$ for some $T > 0$ with initial condition

$$u(0,x) = u_0(x), \quad x \in [0,R], \tag{1.2}$$

and boundary conditions

$$u(t,0) = \mu(t), \quad u(t,R) = \nu(t), \quad u_x(t,R) = h(t), \quad t \in [0,T]. \tag{1.3}$$

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INVERSE PROBLEMS OF FINDING THE LOWER TERM IN A MULTIDIMENSIONAL DEGENERATE PARABOLIC EQUATION

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We study the inverse problem of recovering the lower term of a degenerate parabolic equation with many spatial variables under an additional integral observation condition. We establish sufficient conditions for the unique solvability of the inverse problem in the four cases where the unknown coefficient is looked for in 1) *the space* $L_2(0,T)$, 2) *the class of nonnegative functions in* $L_2(0,T)$, 3) *the space* $L_\infty(0,T)$, 4) *the class of nonnegative functions in* $L_\infty(0,T)$. *Bibliography*: 18 *titles.*

1 Introduction

We study the existence and uniqueness of a solution to the inverse problem of recovering the unknown coefficient $\gamma(t)$ in the parabolic equation

$$u_t - a(t,x)\Delta u + \langle \vec{b}(t,x), u_x\rangle + c(t,x)u + \gamma(t)u = f(t,x) + f^*(t,x), \quad (t,x) \in Q_T, \tag{1.1}$$

with the initial condition

$$u(0,x) = u_0(x), \quad x \in \overline{\Omega}, \tag{1.2}$$

boundary condition

$$u(t,x) = 0, \quad (t,x) \in [0,T] \times \partial\Omega, \tag{1.3}$$

and additional integral observation condition

$$\int\limits_{\Omega} u(t,x)\omega(x)\,dx = \varphi(t), \quad t \in [0,T], \tag{1.4}$$

where $Q_T = [0,T]\times\overline{\Omega}$, Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $x = (x_1,\dots,x_n)$, $\vec{b}(t,x) = (b_1(t,x),\dots,b_n(t,x))$, $u_x = (u_{x_1},\dots,u_{x_n})$, $\langle \vec{b}(t,x), u_x\rangle = \sum\limits_{i=1}^{n} b_i(t,x)u_{x_i}$, $\vec{b}(t,x)$ is a

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APPLICATION OF THE HAUSDORFF METRIC IN MODEL PROBLEMS WITH DISCONTINUOUS FUNCTIONS IN BOUNDARY CONDITIONS

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Using an example of the Cauchy problem for the one-dimensional heat equation, we study the approximation of the solution to the initial condition in the Hausdorff metric. The simplest discontinuous function $u_0(x) = \operatorname{sgn} x$ is taken for the initial condition. Based on the asymptotic behavior of the Lambert W function and its modification, we obtain a two-sided estimate and an asymptotics for the Hausdorff distance between the solution given by the Poisson formula and the function $u_0(x)$. Similar results are obtained for a similar model problem for the Laplace equation in the upper half-plane. Bibliography: 8 *titles.*

1 Introduction

Approximation of discontinuous functions in the Hausdorff metric (H-metric) began to be studied quite a long time ago (see [1, 2]). Such questions arise and are justified in a situation where there is no uniform convergence or there is no L_p-convergence to the limit function, but the graphs are visually close as sets on the plane.

During the discussion at one of our reports, Professor I. V. Tikhonov asked the question whether it is possible to approximate a discontinuous function by the corresponding Poisson integral in the H-metric. This paper is devoted to answering this question.

Some information about the Hausdorff approximation can be found, for example, in [1]–[3]. We recall some necessary definitions.

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NONLINEAR INVERSE PROBLEMS FOR PARABOLIC EQUATIONS WITH TIME–DEPENDENT COEFFICIENTS. REDUCTION TO NONLOCAL PROBLEMS WITH SAMARSKII–IONKIN TYPE CONDITIONS

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We consider coefficient inverse problems of finding a solution and a time-dependent coefficient of a parabolic equation under boundary overdetermination conditions. Based on the very recent results of the first author on nonlocal problems with generalized Samarskii–Ionkin conditions, we establish the solvability of the inverse problems under consideration. Bibliography: 15 *titles.*

We consider coefficient inverse problems of finding a solution $u(x,t)$ and, simultaneously, an unknown coefficient $q(t)$ of the parabolic equation

$$u_t - u_{xx} + q(t)u = f(x,t), \quad x \in (0,1), \quad t \in (0,T), \quad 0 < T < +\infty,$$

with some initial and boundary conditions. Under additional boundary overdetermination conditions, we establish the solvability of the corresponding coefficient inverse problems (Problems I and II below). We note that Problems I and II are related to the class of nonlinear inverse problems because the equation contains the product $q(t)u(x,t)$ of two unknown functions.

In the literature, there are a lot of works devoted to coefficient inverse problems in different statements with different approaches to their study (we refer to some recent results [1]–[7] on inverse problems close to the problems studied in this paper).

In this paper, we are based on the very recent results of the first author [8, 9] (see also [10]) for nonlocal initial-boundary value problems with generalized Samarskii–Ionkin conditions. To

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SOLUTIONS OF AN ILL-POSED STEFAN PROBLEM

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We study the multi-phase Stefan problem with increasing Riemann initial data and generally negative latent specific heats for phase transitions. We propose a variational formulation of self-similar solutions, which allows us to find precise conditions for the existence and uniqueness of a solution. Bibliography: 2 titles.

1 Introduction

In the half-plane $\Pi = \{(t, x) \mid t > 0, x \in \mathbb{R}\}$, we consider the multi-phase Stefan problem for the heat equation

$$u_t = a_i^2 u_{xx}, \quad u_i < u < u_{i+1}, \tag{1.1}$$

where $u_- = u_0 < u_1 < \ldots < u_n < u_{n+1} = u_+$, u_i, $i = 1, \ldots, n$, are the temperatures of phase transitions and $a_i > 0$, $i = 0, \ldots, n$, are the diffusivity constants. On the unknown lines $x = x_i(t)$ of phase transitions, where $u = u_i$, the following Stefan condition is postulated:

$$d_i x_i'(t) + k_i u_x(t, x_i(t)+) - k_{i-1} u_x(t, x_i(t)-) = 0, \tag{1.2}$$

where $k_i > 0$ are the thermal conductivity of the ith phase, whereas d_i is the Stefan number (the latent specific heat) for the ith phase transition. In the case where the temperature $u(t, x)$ is nondecreasing with respect to the spatial variable x, the Stefan numbers d_i should be nonnegative by physical reasons. In this case, the problem (1.1), (1.2) is well-posed and is reduced to a degenerate nonlinear diffusion equation (see [1, Chapter 5]).

In the present paper, we consider the case of arbitrary d_i when the problem is, in general, ill-posed. This fact was first mentioned in [2] for a one-phase Stefan problem with a negative Stefan number. The goal of this paper is to find general conditions guaranteeing the well–posedness of the problem under consideration. We study the Cauchy problem with the Riemann initial data

$$u(0, x) = \begin{cases} u_-, & x < 0, \\ u_+, & x > 0. \end{cases} \tag{1.3}$$

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IMPROVEMENT OF MENSHOV'S THEOREM ON UPPER ESTIMATE FOR THE NORM OF MAXIMAL OPERATOR

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We study the problem of estimating the majorant of partial sums of a series with respect to an orthogonal system. D. E. Menshov established that the norm of the maximal operator does not exceed $\log_2 N + 1$ *and this estimate is order-sharp. We prove that the norm of the maximal operator does not exceed* $0.5 \log_2 N + 1$. *The estimate obtained provides new tools for constructing orthogonal systems with extremely large norm of the majorant of partial sums. Bibliography*: 8 *titles.*

1 Introduction. The Main Result

For an orthonormal system $\{\psi_n\}_{n=1}^{\infty}$ a nondecreasing sequence $\{\omega_n\}_{n=1}^{\infty}$ of nonnegative numbers is called a *Weyl multiplier* for $\{\psi_n\}_{n=1}^{\infty}$ if for any coefficient sequence $\{c_n\}_{n=1}^{\infty}$, $\sum\limits_{n=1}^{\infty} c_n^2 \omega_n < +\infty$, the series $\sum\limits_{n=1}^{\infty} c_n \psi_n(x)$ converges almost everywhere. The classical Rademacher–Menshov theorem (see [1]–[3]) asserts that the sequence $\omega_n = \log_2^2(n+1)$ is the exact Weyl multiplier on the class of all orthonormal systems. In other words, for any orthonormal system $\{\psi_n\}_{n=1}^{\infty}$ the condition

$$\sum_{n=1}^{\infty} c_n^2 \log_2^2(n+1) < +\infty$$

implies the convergence of $\sum\limits_{n=1}^{\infty} c_n \psi_n(x)$ almost everywhere. Conversely, there exists an orthonormal system $\{\psi_n\}_{n=1}^{\infty}$ such that for any nondecreasing sequence $\omega_n = o(\log_2^2(n+1))$ of nonnegative numbers there exists a coefficient sequence $\{c_n\}_{n=1}^{\infty}$ such that $\sum\limits_{n=1}^{\infty} c_n^2 \omega_n < +\infty$ and the series $\sum\limits_{n=1}^{\infty} c_n \psi_n(x)$ diverges almost everywhere. The above results are based on the following theorems proved by Menshov (see [3]).

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SYMMETRIES OF FRACTIONAL GUÉANT–PU MODEL WITH GERASIMOV–CAPUTO TIME-DERIVATIVE

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For the time fractional Guéant–Pu option pricing model we obtain the Lie algebra of the group of equivalence transformations, which is used to obtain the group classification of the model with respect to the nonlinear free element on the right-hand side of the corresponding equation. Bibliography: 12 *titles.*

1 Introduction

We study a Guéant–Pu option pricing model (see arXiv:1311.4342 and [1]) that takes into account transaction costs and the impact of operations on the market in the absence of a permanent market impact. We introduce the notation: $r \in \mathbb{R}$ is the risk-free rate, $\gamma \in \mathbb{R}$ is the absolute risk aversion parameter, $\sigma > 0$ is the volatility, and $\mu \in \mathbb{R}$ is the trend forecast expected return of the underlying asset. The indifference price of a call option $\theta = \theta(t, S, q)$ depends on time t, the price of underlying asset S, and the number of shares in the hedged portfolio q.

We consider the Guéant–Pu model in the form

$$\theta_t = r\theta + (\mu - rS)q - \mu\theta_S - \frac{\sigma^2}{2}\theta_{SS} - \frac{1}{2}\gamma\sigma^2 e^{r(T-t)}(\theta_S - q)^2 + F(t, \theta_q). \tag{1.1}$$

Equation (1.1) was studied by methods of symmetry analysis in [2]–[6], where the group classification was obtained and invariant solutions were calculated. In particular, the cases $F = F(\theta_q)$ and $F = F(t, \theta_q)$ were considered in [2, 3] and [4, 5] respectively, whereas recursion operators for Equation (1.1) with $F = a(t)\theta_q$ were constructed in [6].

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VOLUME POTENTIAL AND INITIAL–BOUNDARY VALUE PROBLEMS FOR PARABOLIC SYSTEMS WITH DINI CONTINUOUS COEFFICIENTS IN THE PLANE

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We study the smoothness in the Dini space of a vector parabolic volume potential whose density can be unbounded near the parabolic boundary of the domain. We establish the solvability of initial-boundary value problems for inhomogeneous parabolic systems with Dini continuous coefficients in a semibounded domain on the plane. We prove estimates for solutions. Bibliography: 10 *titles.*

In this paper, we study the smoothness of a vector parabolic volume potential generated by the fundamental matrix of solutions to a one-dimensional Petrovskii second order parabolic system with Dini continuous coefficients. The density f of the potential is distributed in a curvilinear domain on the plane with nonsmooth lateral boundary. We assume that f can grow to infinity in a certain way when approaching the parabolic boundary of the domain. We prove estimates in the Dini space for the potential and its spatial first order derivative. This result is new even in the case of a single equation. Similar estimates in the Hölder space were obtained in [1] in the case of a multidimensional parabolic equation with coefficients from the Hölder class. In the case of a one-dimensional parabolic equation with Dini continuous coefficients, estimates for a volume potential with a density f distributed in a strip are established in [2, 3] under the condition that f is bounded, and in [4] under the condition that f can grow in a certain way as $t \to +0$. As a consequence, we establish the solvability of initial-boundary value problems for a one-dimensional inhomogeneous parabolic system with Dini continuous coefficients such that its right-hand side is not necessarily bounded near the parabolic boundary of the domain. We also obtain estimates for the solutions and their spatial first order derivatives.

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