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VARIATIONS OF STATEMENT, VARIATIONS OF STRENGTH. THE CASE OF THE RIVAL–SANDS THEOREMS

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The paper surveys results obtained mainly by the author with A. Marcone, P. Shafer, and G. Soldà centered around two theorems due to I. Rival and B. Sands. We analyze the strength of these theorems and their variations within the framework of reverse mathematics and Weihrauch reducibility. Some related combinatorial results are presented, as well as generalizations of these results to the uncountable. Open questions are formulated. Bibliography: 17 *titles. Illustrations*: 2 *figures.*

1 Introduction

Ramsey's theorem for pairs, in its graph-theoretic formulation, guarantees that each infinite undirected graph contains an infinite subgraph which is either totally disconnected, i.e., the elements of the subgraph are pairwise nonadjacent or complete, i.e., the elements of the subgraph are pairwise adjacent. Ramsey's theorem for pairs, thus, presents a perfect dichotomy between two possible adjacency structures a solution can have. Nevertheless, the solutions are totally forgetful of the structure of the remaining part of the graph. From this observation Rival and Sands took inspiration to prove the following theorem [1].

Theorem 1.1. *Every infinite undirected graph G contains an infinite subset H such that every vertex of G is adjacent to precisely none, one, or infinitely many vertices of H.*

In this paper, we often use the terminology "instance" and "solution" of a theorem. To explain them through an example, any undirected graph G is an instance of the previous theorem and any H satisfying the conclusion of the theorem is a solution.

The solution of Theorem 1.1 is required to satisfy some constrains on the adjacency relation between vertices which belong to the solution and vertices which do not (this is why we referred to this theorem as an inside/outside Ramsey's theorem in [2]). In this respect, the previous theorem can be seen as an improvement of Ramsey's theorem for pairs. However, the perfect dichotomy of the solutions of Ramsey's theorem for pairs is lost. Indeed, Rival and Sands

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CLASSES OF ALGEBRAIC STRUCTURES

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This paper is a survey of some topics in computable structure theory, illustrating the power of certain infinitary sentences to describe mathematical structures and classes of structures. It includes classical results such as the Scott isomorphism theorem and the Lopez-Escobar theorem, and work of Friedman and Stanley on comparing classes of structures according to the complexity of their invariants. It also includes more recent results, in particular, on torsion-free Abelian groups. Bibliography: 25 titles. Illustrations: 1 figure.

1 Conventions

We are interested in countable structures and classes of structures. We observe the following conventions.

- Structures have universe ω or a subset.
- Languages are countable, usually computable.
- Classes of structures have fixed language and are closed under isomorphism.

2 The Logic $L_{\omega_1,\omega}$

For the infinitary logic $L_{\omega_1,\omega}$, formulas have countably infinite disjunctions and conjunctions, but only finite strings of quantifiers.

2.1. Sample formulas. The following sentence says of a real closed ordered field that it is Archimedean:

$$(\forall x) \bigvee_n x < \underbrace{1 + \cdots + 1}_n .$$

The following formula says of an element in an Abelian group that it is not torsion:

$$\bigwedge_n \underbrace{x + \cdots + x}_n \neq 0.$$

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CALCULATION OF THE VALUES OF THE RIEMANN ZETA FUNCTION VIA VALUES OF ITS DERIVATIVES AT A SINGLE POINT

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We consider the following two problems. We are given the values of several initial derivatives of the Riemann zeta function calculated at some (unknown to us) point a.

- *How could we calculate an approximate value of the function itself at the same point a without prior finding this number?*
- *How could we find an approximate value of a itself?*

We suggest several algorithms for answering these questions and demonstrate their accuracy on a few numerical examples. The algorithms reveal some new properties of the zeta function. Bibliography: 9 titles.

1 The Riemann Zeta Function

This section presents some known facts required for the rest of the paper.

Prime numbers 2, 3, ... are one of the most important objects of investigations in Number Theory. An important and efficient tool for studying them is the celebrated *Riemann zeta function*. For a complex number s such that $\mathrm{Re}\,(s) > 1$ it can be defined by a *Dirichlet series*, namely,

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}. \tag{1.1}$$

This function was studied already by L. Euler (for real values of s only). Besides $\zeta(s)$, he considered the *alternating zeta function* (known also as the *Dirichlet eta function*),

$$\eta(s) = \sum_{n=1}^{\infty} (-1)^{n+1} n^{-s}. \tag{1.2}$$

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SUBLINKS OF STRONGLY QUASIPOSITIVE LINKS

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We prove that any two given links can be combined to give a strongly quasipositive link, which implies that any link is a sublink of a strongly quasipositive link. We discuss some complexity issues of the strongly quasipositive link constructed. Bibliography: 25 *titles.*

1 Introduction, Motivation and the Main Result

The notion of a Bennequin surface, so named by Birman–Menasco [1], originates from Bennequin's work [2] and refers to a braided Seifert surface of minimal genus. Rudolph [3] has shown that every Seifert surface can be made into a braided form, so that Bennequin surfaces exist for every link. These surfaces are closely related to (and particularly important for) strongly quasipositive links.

Various notions of positivity of links have been studied also with motivation outside the field of knot theory. If the zero set of a complex polynomial $f : \mathbb{C}^2 \to \mathbb{C}$ intersects the unit sphere $S^3 = \{ (u,v) \in \mathbb{C}^2 : |u|^2 + |v|^2 = 1 \}$ transversely, then the intersection forms a link in S^3. By work of Rudolph [4] and Boileau–Orevkov [5] it was proved that these links are precisely the quasipositive links.

A link is called *quasipositive* if it is the closure of a braid β of the form

$$\beta = \prod_{j=1}^{n} w_j \sigma_{i_j} w_j^{-1},$$

where w_j is any braid word and σ_{i_j} is a (positive) standard Artin generator of the braid group. (In [6] there is some overview of this topic.) If the words $w_j \sigma_{i_j} w_j^{-1}$ are of the form [1)]

$$\sigma_{i,j} = \sigma_i^{-1} \dots \sigma_{j-2}^{-1} \sigma_{j-1} \sigma_{j-2} \dots \sigma_i, \tag{1.1}$$

then they can be regarded as embedded bands (see the left hand-side of (3.5)). Links which arise this way are called *strongly quasipositive links*, and it was proved that they contain the class of *positive* links, i.e., links with diagrams all of whose crossings are positive (right-hand) [7].

[1)] Note that thus the bands are, from our perspective, behind the Seifert disks put into the braid strands to obtain a strongly quasipositive Seifert surface. We will maintain this convention throughout the paper.

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ISOLATION FROM SIDE AND CONE AVOIDANCE IN THE 2-COMPUTABLY ENUMERABLE *wtt*-DEGREES

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We consider isolation from side in the structure of 2*-computably enumerable wtt-degrees. Intuitively, a* 2*-computably enumerable degree* **d** *is isolated from side if all computably enumerable degrees from its lower cone are bounded by some computably enumerable degree incomparable with* **d**. *We prove that any proper* 2*-computably enumerable wtt-degree is isolated from side by some computable enumerable wtt-degree. We show how this property can be combined with a cone avoidance. Bibliography*: 14 *titles.*

1 Introduction

A short version of this work with basic ideas of the main proof was published in [1]. In this paper, we give the detailed proofs as well as consider more general results. The work is devoted to an investigation of structural properties of computably enumerable (further, c.e.) and 2-c.e. degrees of unsolvability. We focus on *wtt*-degrees and study a concept of *isolation from side*, which is a variant of isolation for 2-c.e. degrees. Isolated 2-c.e. Turing degrees were introduced by B. Cooper and X. Yi (Preprint, Univ. Leeds, 1995). They proved the existence of isolated 2-c.e. Turing degrees. Recall that a 2-c.e. Turing degree **d** is isolated if there exists a c.e. Turing degree $\mathbf{c} < \mathbf{d}$ such that there is no a c.e. Turing degree strictly between **d** and **c**. It is easy to see that the degree **c** is the greatest c.e. Turing degree below **d**. Note that, by the Sacks Density Theorem [2], the concept of isolation can be considered only for proper 2-c.e. degrees (i.e., for 2-c.e. degrees that do not contain c.e. sets).

Isolated 2-c.e. Turing degrees found out to be useful in series of constructions (see, for example, [3]) and allowed one to employ the properness of considered 2-c.e. degrees. Unfortunately, it turned out that non-isolated proper 2-c.e. Turing degrees exist too, and, moreover, such degrees are as “common” as isolated ones (see [4]–[6]). Thus, the isolation property cannot characterize proper 2-c.e. Turing degrees. However, such characterization followed by interpretation of c.e. Turing degrees in the structure of 2-c.e. Turing degrees in the language of partial ordering solves in a natural way the well-known problem of definability of c.e. Turing degrees in the 2-c.e. Turing degrees (the problem was stated in series of works, for example, in [7] and [8]).

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THE ORBITAL STABILITY ANALYSIS OF PENDULUM OSCILLATIONS OF A HEAVY RIGID BODY WITH A FIXED POINT UNDER THE GORIACHEV–CHAPLYGIN CONDITION

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We consider the motion of a heavy rigid body with a fixed point in a uniform gravitational field under the assumption that the principal moments of inertia satisfy the Goryachev–Chaplygin condition at the fixed point. We study the orbital stability problem for small pendulum oscillations of the body. We derive the equations of perturbed motion and reduce the problem to the study of the stability of the equilibrium position of a second order 2π*-periodic Hamiltonian system. We find regions of parametric resonance and perform the nonlinear analysis of orbital stability outside these regions. Bibliography*: 14 *titles.*

Periodic motions play a special role in rigid body dynamics. The study of such motions often makes it possible to draw important conclusions on motion properties of a considered mechanical system, as well as it helps to perform qualitative analysis of the phase space of the system. By this reason, the problem of orbital stability of pendulum periodic motions of a heavy rigid body with a fixed point is of considerable interest for both theoretical mechanics and its applications. Modern methods of the theory of dynamical systems, including the method of normal forms, the methods of the Kolmogorov–Arnold–Moser theory and general theory of stability allow one to obtain rigorous conclusions about the orbital stability of periodic motions of this type.

In the general case, the problem of orbital stability of pendulum periodic motions of a heavy rigid body with one fixed point contains four parameters. To reduce the number of parameters in the problem, the most interesting special cases are usually considered. The Kovalevskaya

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BOUNDS FOR THE CLUSTERING COMPLEXITY IN A GRAPH CLUSTERING PROBLEM WITH CLUSTERS OF BOUNDED SIZE

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We consider the graph clustering problem under the assumption that the size of each cluster is bounded by a given positive integer s. In the case $s = 4$, we prove that the clustering complexity of an arbitrary n-vertex graph, where $n \geqslant 5$, does not exceed $n(n-1)/2 - 6\lfloor n/4 \rfloor$. As a consequence, we obtain the same upper bound for the clustering complexity of a graph in the general case $s \geqslant 4$. Bibliography: 2 titles.

1 Statement of the Problem

We consider the graph clustering problem in the case where each cluster is bounded from above by the same positive integer. We deal with ordinary graphs, i.e., graphs without loops and multiple edges. An ordinary graph $G = (V, E)$ with a vertex set V is called a *cluster graph* if every connected component of G is a complete graph [1]. The *distance* $d(G_1, G_2)$ between ordinary graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ on the set of vertices V is defined by

$$d(G_1, G_2) = |E_1 \Delta E_2| = |E_1 \setminus E_2| + |E_2 \setminus E_1|,$$

i.e., $d(G_1, G_2)$ is the number of noncoinciding edges in G_1 and G_2. We denote by $\mathscr{M}^{\leqslant s}(V)$ the family of all cluster graphs on V such that the size of each connected component is at most s, where s is an integer such that $2 \leqslant s \leqslant |V|$.

Problem $\mathrm{CE}^{\leqslant s}$**.** For a graph $G = (V, E)$ and an integer $2 \leqslant s \leqslant |V|$ find $M^* \in \mathscr{M}^{\leqslant s}(V)$ such that

$$d(G, M^*) = \min_{M \in \mathscr{M}^{\leqslant s}(V)} d(G, M) \overset{dn}{=} \tau^{\leqslant s}(G).$$

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ON THE CROSS LEMMAS IN ONE-SIDED SEMI- AND QUASI-ABELIAN CATEGORIES

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We study the validity of the so-called cross lemmas in one-sided semi- and quasi-abelian categories. Bibliography: 15 *titles.*

1 Introduction

The so-called cross lemmas play an important part in the theory of various classes of additive categories (see, for example, [1]–[3]).

In its typical statement, a cross lemma is an assertion of the following form in a pre-Abelian category (i.e., an additive category with kernels and cokernels): Assume that, in the commutative diagram

$$\begin{array}{ccccc} & & D & & \\ & & \downarrow{\scriptstyle f} & & \\ A & \xrightarrow{\beta} & B & \xrightarrow{p} & E \\ & & \downarrow{\scriptstyle q} & & \\ & & F & & \end{array}$$

in a pre-Abelian category, $\mathrm{im}\beta = \ker p$, $\mathrm{im} f = \ker q$, the morphism $q\beta$ is strict, and some additional conditions are fulfilled. Then the composition pf is also strict.

A cross lemma was used in [1, Lemma 7]) to prove that (in the modern terminology) a left- or right quasi-Abelian semi-Abelian category is, in fact, quasi-Abelian.

A more general cross lemma was established in a quasi-Abelian category in [2, Lemma 9] and was used there as a tool for proving the strictness of some morphisms in the Ker-Coker-sequence in a quasi-Abelian category.

A general theorem, called the *square-cross lemma*, was studied in a Quillen exact category in [3], where it was proved that, in an Abelian category, the diagrams of the kind involved in the theorem are in an one-to-one correspondence with cross diagrams of the form

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ON A CYLINDER FREELY FLOATING IN OBLIQUE WAVES

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We investigate the coupled motion of a mechanical system consisting of water and a body freely floating in it. The water occupies either a half-space or a layer of constant depth into which an infinitely long surface-piercing cylinder is immersed, thus allowing us to study the so-called oblique waves. Under the assumption that the motion is of small amplitude near equilibrium and describes time-harmonic oscillations, the linear setting of the phenomenon reduces to a spectral problem with the radian frequency as the spectral parameter. If the radiation condition is fulfilled, then the total energy is finite and the equipartition of kinetic and potential energy takes place for the whole system. On this basis, it is proved that no wave modes are trapped under some restrictions on their frequencies. In the case where a symmetric cylinder has two immersed parts, restrictions are imposed on the type of mode as well. Bibliography: 12 *titles. Illustrations*: 1 *figure.*

1 Introduction

This paper continues the author's studies dealing with the motion of a mechanical system consisting of a water layer of constant depth and a rigid body freely floating in it. The initial publication [1] was written more than ten years ago and several papers on this topic appeared since then (see, for example, [2]–[4]). It was John [5] who proposed the linear problem describing the coupled motion of water bounded from above by the atmosphere and a partially immersed body. The latter floats freely according to Archimedes' law being unaffected by all external forces (for example, due to constraints on its motion) except for gravity. The motion of water (its viscosity is neglected as well as the surface tension on its surface) is assumed to be irrotational, whereas the motion of the whole system is supposed to be of small amplitude near equilibrium; this allows us to use a linear mathematical model.

Within the framework of the linear theory of water waves, two- and three-dimensional formulations are possible. The problem considered here is two-dimensional which is essential for formulating the geometric and other restrictions that must be imposed to guarantee the uniqueness. The latter is of paramount importance (in the classical survey [6], it is placed at the top of the list of important open problems) because there are examples of nonuniqueness (see,

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EXPERIMENTAL STUDY OF SEMI-SUPERVISED GRAPH 2-CLUSTERING PROBLEM

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Based on results of experiments, we analyze the efficiency of some exact and approximation algorithms for a semi-supervised graph 2*-clustering problem. Bibliography*: 6 *titles. Illustrations*: 3 *figures.*

1 Introduction

In clustering problems it is required to split, based on similarity of objects, the set of objects into subsets (clusters). The clustering problems form an important part of machine learning and are usually classified as unsupervised learning. However, semi-supervised algorithms and methods are also applicable [1] to clustering problems. Under this approach, only a few objects are labeled, whereas a large number of objects remain unlabeled. In graph clustering problems, the set of objects consists of vertices and the similarity of objects is defined through the edges of the graph.

In this paper, we deal only with simple graphs, i.e., undirected graphs without loops and multi-edges. We say that a graph is a *cluster graph* if each of its components is a clique. The *distance* $\rho(G_1, G_2)$ between two labeled graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ is the cardinality of the symmetric difference $E_1 \Delta E_2$ (elements of this set are called *disagreements*):

$$\rho(G_1, G_2) = |E_1 \Delta E_2| = |E_1 \setminus E_2| + |E_2 \setminus E_1|. \tag{1.1}$$

Hereinafter, we denote by V the set of vertices and by E the set of edges of a graph G and write $G = (V, E)$. For a vertex v of a graph $G = (V, E)$ by its neighborhood $N_G(v)$ we mean the set of $u \in V$ joined with the vertex v. We formulate the NP-hard graph clustering problem in the semi-supervised learning case.

Problem 1.1 (k-semi-supervised graph clustering $[p_1, \dots, p_k]$)**.** Assume that a graph $G = (V, E)$ in an input, $2 \leqslant k \leqslant |V|$ is an integer, and $\mathscr{Z} = \{Z_1, \dots, Z_k\}$ is a collection of pairwise disjoint nonempty subsets of V such that $|Z_1| = p_1, \dots, |Z_k| = p_k$. Find a cluster graph C with k clusters minimizing the number of disagreements. In addition, the sets in $\mathscr{Z}$ must belong to different clusters of C.

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